A NOTE ON NILPOTENT REPRESENTATIONS

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Abstract. Let Γ be a finitely generated nilpotent group and let G be a complex reductive algebraic group. The representation variety $\mathrm{Hom}(\Gamma, G)$ and the character variety $\mathrm{Hom}(\Gamma, G)/\!\!/G$ each carry a natural topology, and we describe the topology of their connected components in terms of representations factoring through quotients of Γ by elements of its lower central series.

1. Introduction

Let G be the group of complex points of an affine algebraic group. When Γ is a finitely generated group, one may parametrize the homomorphisms from Γ to G by the images of a finite generating set. This realizes $\mathrm{Hom}(\Gamma, G)$ as an (affine) algebraic set, carved out of a finite product of copies of G by the relations of Γ. As a complex variety, $\mathrm{Hom}(\Gamma, G)$ admits a natural Hausdorff topology obtained from an embedding into affine space and it is easy to see (and well-known) that the analytic space structure on $\mathrm{Hom}(\Gamma, G)$ is independent of the chosen presentation of Γ. Here, we will only consider the case where G is reductive though, in principle, the questions we address below can be asked without this assumption.

These spaces of homomorphisms are of classical interest (see Lubotzky–Magid [17] and the references therein) and their algebraic topology has been the subject of much recent scrutiny (see, for instance, [2, 3, 4, 5, 11, 12, 15]), stemming in part from the work of Ádem and Cohen [1]. In this context, it was recently shown by the first named author [6] that if Γ is nilpotent and K is a maximal compact subgroup of G, then there is a strong deformation retraction of $\mathrm{Hom}(\Gamma, G)$ onto $\mathrm{Hom}(\Gamma, K)$. This result was first established by homotopy-theoretic methods for Γ abelian by Pettet and Souto [19] and for Γ expanding nilpotent by Souto and the second named author. The result for arbitrary nilpotent groups was obtained in [6] by replacing these earlier approaches with algebro-geometric methods. Nevertheless, the machinery developed by Pettet–Souto and its followups is very well posed to the study of topological invariants. Accordingly, the goal of this note is to combine

2010 *Mathematics Subject Classification.* Primary 55P99; Secondary 20F18, 20C99, 20G20.

Key words and phrases. Representation variety, nilpotent groups, central series.

M.B. was supported by an NSERC Alexander Graham Bell CGS-D Scholarship.

L.S. was partly supported by an NSERC Discovery Grant.

these topological and algebro-geometric tools to obtain topological information about representation spaces of nilpotent groups.

From now on, fix a non-abelian finitely generated s-step nilpotent group Γ. Recall that this means that the lower central series, defined inductively by

$$\Gamma_{(1)} = \Gamma, \qquad \Gamma_{(i+1)} = [\Gamma, \Gamma_{(i)}]$$

has $\Gamma_{(s)}$ non-trivial but $\Gamma_{(s+1)} = \{e\}$. The epimorphism $\Gamma \to \Gamma/\Gamma_{(i)}$ induces an embedding

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, G) \to \mathrm{Hom}(\Gamma, G)$$

which (for general groups Γ and G) is not even an open map. Nevertheless, we will show:

Theorem 1.1. *Let Γ be a finitely generated nilpotent group and let G be the group of complex points of a (possibly disconnected) reductive algebraic group. For all $i \geq 2$, the inclusion*

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, G) \xrightarrow{\iota} \mathrm{Hom}(\Gamma, G)$$

is a homotopy equivalence onto the union of those components of the target intersecting the image of ι.

Consider $\mathrm{Hom}(\Gamma, G)$ as a based space by taking the trivial representation as the base point. In this case, Theorem 1.1 implies that the connected components

$$\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \subset \mathrm{Hom}(\Gamma, G) \text{ and } \mathrm{Hom}(\Gamma/\Gamma_{(i)}, G)_{\mathbb{1}} \subset \mathrm{Hom}(\Gamma/\Gamma_{(i)}, G)$$

of the trivial representation are homotopy equivalent for all $i \geq 2$. Using this, we will describe the homotopy type of the component of the trivial representation in terms of abelian representations:

Corollary 1.2. *For Γ and G as in* Theorem 1.1*, there is a homotopy equivalence*

$$\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \simeq \mathrm{Hom}(\mathbb{Z}^{\operatorname{rank} H_1(\Gamma;\mathbb{Z})}, G)_{\mathbb{1}}.$$

To introduce the other space we study, note first that the action of G on itself by conjugation induces an action on $\mathrm{Hom}(\Gamma, G)$ and conjugate homomorphisms are often considered equivalent (this is the usual notion of equivalence of representations in $\mathrm{GL_n}\mathbb{C}$). Accordingly one often wishes to understand the associated quotient but, unfortunately, the naive topological quotient is not a nice space: it need not even be Hausdorff. In order to "repair" this space, we use the affine geometric invariant theory quotient $\mathrm{Hom}(\Gamma, G)/\!\!/G$ instead. This so-called *character variety* is usually endowed with the structure of an affine variety but, for our purposes, it may be constructed topologically as the universal quotient in the category of Hausdorff spaces (see Brion–Schwarz [10]). The systematic study of the topology of theses spaces has seen much recent development (see, for instance, [7, 8, 13, 14, 16]). Concentrating on the component of the trivial representation, we will use Corollary 1.2 to prove:

Corollary 1.3. *Let Γ be a finitely generated nilpotent group and let G be the group of complex points of a reductive algebraic group. Then*

(1) $\pi_1\left(\mathrm{Hom}(\Gamma, G)_{\mathbb{1}}\right) \cong \pi_1(G)^{\,\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}$, *and*
(2) $\pi_1\left((\mathrm{Hom}(\Gamma, G)/\!\!/ G)_{\mathbb{1}}\right) \cong \pi_1(G/[G,G])^{\,\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}$.

Corollary 1.4. *Let G be the group of complex points of a connected reductive algebraic group, let $T \subset G$ be a maximal algebraic torus and let W be the Weyl group of G. If Γ is a finitely generated nilpotent group and F is a field of characteristic 0 or relatively prime to the order of W, then:*

(1) $H^*\left(\mathrm{Hom}(\Gamma, G)_{\mathbb{1}}; F\right) \cong H^*(G/T \times T^{\,\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}; F)^W$, *and*
(2) $H^*\left((\mathrm{Hom}(\Gamma, G)/\!\!/ G)_{\mathbb{1}}; F\right) \cong H^*(T^{\,\mathrm{rank}\, H_1(\Gamma;\mathbb{Z})}; F)^W$.

While the results above indicate many similarities between representation spaces of abelian and non-abelian nilpotent groups, the latter have a much richer topology than the former. For instance, recall that for a connected semisimple group S, the variety $\mathrm{Hom}(\mathbb{Z}^2, S)$ is irreducible and thus connected [20]. Moreover, $\mathrm{Hom}(\mathbb{Z}^r, \mathrm{SL}_n\,\mathbb{C})$, $\mathrm{Hom}(\mathbb{Z}^r, \mathrm{Sp}_{2n}\,\mathbb{C})$ and the corresponding character varieties are connected for all values of r and n. The situation for non-abelian nilpotent groups is markedly different:

Theorem 1.5. *Let G be the group of complex points of a (possibly disconnected) reductive algebraic group. If Γ is a finitely generated nilpotent group which surjects onto a finite non-abelian subgroup of G, then* $\mathrm{Hom}(\Gamma, G)$ *and* $\mathrm{Hom}(\Gamma, G)/\!\!/ G$ *are both disconnected topological spaces.*

Since non-abelian free nilpotent groups and Heisenberg groups surject onto the non-abelian nilpotent group of order 8, this implies:

Corollary 1.6. *Let Γ be a non-abelian free nilpotent group or a Heisenberg group. If G is the group of complex points of a reductive algebraic group, then* $\mathrm{Hom}(\Gamma, G)$ *and* $\mathrm{Hom}(\Gamma, G)/\!\!/ G$ *are connected if and only if G is an algebraic torus.*

Remark. All of the preceding statements remain true when G is replaced by a compact Lie group K. In fact, we will prove most of them in this setting before obtaining the complex reductive case via a homotopy equivalence.

Outline of the paper. We begin Section 2 by describing compact representation spaces using a fibre bundle. Then, in Section 3, we use this bundle to prove Theorem 1.1 and Theorem 1.5 along with their various corollaries.

Acknowledgements. The authors would like to thank Alejandro Ádem and Alexandra Pettet for their comments on a preliminary version of this note. The first named author would also like to thank Man Chuen Cheng, Justin Martel and Juan Souto for many stimulating conversations.

2. An interesting bundle

The goal of this section is to prove the following key proposition:

Proposition 2.1. *Let K be a (possibly disconnected) compact Lie group. If Γ is an s-step nilpotent group with $s \geq 2$, then the set of abelian groups*

$$\mathcal{F} := \{\rho(\Gamma_{(s)}) \subset K : \rho \in \mathrm{Hom}(\Gamma, K)\}$$

admits a homogeneous manifold structure with finitely many connected components for which the projection map

$$p : \mathrm{Hom}(\Gamma, K) \to \mathcal{F},\ p(\rho) = \rho(\Gamma_{(s)}) \tag{1}$$

is a locally trivial fibre bundle.

The proof of Proposition 2.1 relies on the following lemma:

Lemma 2.2. *For all $m \in \mathbb{N}$ there is an $O = O(m) \in \mathbb{N}$ such that, if $N \subset \mathrm{SU}_m$ is an s-step nilpotent group with $s \geq 2$, then $N_{(s)}$ is an abelian subgroup of SU_m of order bounded by O.*

Proof. Recall that $N_{(s)}$ is an abelian subgroup of SU_m contained in the centre of N. As such, there is a direct sum decomposition $\mathbb{C}^m = V_1 \oplus \ldots \oplus V_r$ and r characters $\chi_1, \ldots, \chi_r : N_{(s)} \to \mathbb{C}^\times$ such that $\chi_i \neq \chi_j$ for all $i \neq j$ and $\gamma(v) = \chi_i(\gamma) \cdot v$ for all $\gamma \in N_{(s)}$ and $v \in V_i$. Moreover, for all $g \in N$, $\gamma \in N_{(s)}$ and $v \in V_i$, we have

$$\gamma(g(v)) = g(\gamma(v)) = g(\chi_i(\gamma) \cdot v) = \chi_i(\gamma) \cdot g(v).$$

This allows us to consider the restrictions of the determinant homomorphism

$$\det_i : N \to \mathbb{C}^\times,\ \ \det_i(g) := \det(g|_{V_i})$$

where, since $\mathbb{C}^\times$ is abelian and $s \geq 2$, the subgroup $N_{(s)}$ must be contained in $\ker(\det_i)$. This means that for all i and all $\gamma \in N_{(s)}$, we have

$$\det_i(\gamma) = \chi_i(\gamma)^{\dim V_i} = 1,$$

so $\chi_i(\gamma)$ is always a root of unity of order bounded by m. Consequently, $N_{(s)}$ is conjugate in SU_m to a subset of those diagonal matrices whose diagonal elements are roots of unity of order bounded by m. This completes the proof since the order of this finite set does not depend on s. □

Proof of Proposition 2.1. Choose a faithful embedding of K into SU_m. By Lemma 2.2, there is a constant $O \in \mathbb{N}$ uniformly bounding the order of abelian subgroups of K occurring as the image of $\Gamma_{(s)}$ under homomorphisms $\rho : \Gamma \to K$. In order to give $\mathcal{F}$ a homogeneous manifold structure, we first consider the slightly larger set

$$\tilde{\mathcal{F}} := \{A \subset K : A \text{ is an abelian subgroup of order bounded by } O\}.$$

Observe that K^o (the identity component of K) acts by conjugation on $\tilde{\mathcal{F}}$ with closed stabilizers. As such, we can endow $\tilde{\mathcal{F}}$ with the orbifold structure with respect to which each K^o-orbit is a connected homogeneous K^o-manifold (see [18]). Concretely, if we define the "connected normalizer" as $N_{K^o}(H) := N_K(H) \cap K^o$, then the connected component of $H \in \tilde{\mathcal{F}}$ is identified with $K^o/N_{K^o}(H)$. Having a topology on each K^o-orbit, we endow $\tilde{\mathcal{F}}$ with the disjoint union topology. Since K is a compact Lie group, there are only finitely many conjugacy classes of abelian subgroups of K of order bounded by O and, in particular, $\tilde{\mathcal{F}}$ has only finitely many connected components.

A homomorphism $\rho : \Gamma_{(s)} \to K$ need not extend to the full group Γ so the map

$$p : \mathrm{Hom}(\Gamma, K) \to \tilde{\mathcal{F}},\ p(\rho) = \rho(\Gamma_{(s)})$$

may not be surjective. Accordingly, we denote $\mathcal{F} := p(\mathrm{Hom}(\Gamma, K))$ and observe by K^o-equivariance of p that it is a union of connected components of $\tilde{\mathcal{F}}$. Let $\mathcal{Z} \subset \mathcal{F}$ denote the connected component of a finite abelian subgroup $H \in \mathcal{F}$ and let $\mathcal{H} := p^{-1}(\mathcal{Z}) \subset \mathrm{Hom}(\Gamma, K)$. Since $\mathcal{F}$ has only finitely many components, it follows that p is a continuous map and it now suffices to show that $p\colon \mathcal{H} \to \mathcal{Z}$ is a locally trivial fibre bundle. Observing once again that p is K^o- equivariant, this follows at once from [9, Proposition 2.3.2]. More concretely, letting $\mathcal{H}(H) := p^{-1}(H)$, we can identify the restriction of p to $\mathcal{H}$ with the twisted product

$$(K^o \times \mathcal{H}(H))/N_{K^o}(H) \to K^o/N_{K^o}(H)$$

where $N_{K^o}(H)$ acts on K^o (resp. $\mathcal{H}(H)$) by right multiplication (resp. conjugation). □

3. Proofs of the main results

Let G be the group of complex points of a (possibly disconnected) reductive algebraic group and recall that such a G necessarily arises as the complexification of a (possibly disconnected) compact Lie group K. In this section, we use Proposition 2.1 to prove the results mentioned in the introduction. In most cases, we prove a corresponding statement with K in lieu of G before obtaining the claimed result. We refer the reader to Onishchick–Vinberg [18] for basic facts about Lie groups and complex algebraic groups.

Let Γ be an s-step nilpotent group with $s \geq 2$ and recall that, for all i, the epimorphism $\Gamma \to \Gamma/\Gamma_{(i)}$ induces an embedding $\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K) \to \mathrm{Hom}(\Gamma, K)$. Often, we shall abuse notation and identify $\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K)$ with its image under this embedding. As a first consequence of Proposition 2.1 we obtain:

Proposition 3.1. *Let K be a (possibly disconnected) compact Lie group. If Γ is a finitely generated nilpotent group then, for all $i \geq 2$, the inclusion*

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K) \xrightarrow{\iota} \mathrm{Hom}(\Gamma, K)$$

is a homeomorphism onto the union of those components of the target intersecting the image of ι.

Proof. We proceed by induction on the nilpotence step of Γ. Recall from Proposition 2.1 that

$$p : \mathrm{Hom}(\Gamma, K) \to \mathcal{F},\, p(\rho) = \rho(\Gamma_{(s)})$$

is a locally trivial bundle. If Γ is 2-step nilpotent, then the image of ι consists of all representations factoring through the abelianization of Γ, that is those such that $\rho(\Gamma_{(2)}) = \{e_K\}$. Since e_K is fixed by the conjugation action of K, the subgroup $\{e_K\} \in \mathcal{F}$ is an isolated point in the given topology. Thus, for any $\rho \in p^{-1}(e_K)$, the full connected component of ρ (which is path-connected) has trivial restriction to $\Gamma_{(2)}$ and we see that $p^{-1}(\{e_K\})$ is the union of the connected components it intersects, completing the proof in this case.

Suppose now that Γ is s-step nilpotent. If $i = s$, the same argument as for the base case applies. Otherwise, $i < s$ and then

$$\Gamma/\Gamma_{(i)} \cong (\Gamma/\Gamma_{(s)})/(\Gamma_{(i)}/\Gamma_{(s)})$$

where the nilpotence step of $(\Gamma/\Gamma_{(s)})$ is $s-1$. As such, the induction hypothesis implies that each of the following two embeddings

$$\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K) \to \mathrm{Hom}(\Gamma/\Gamma_{(s)}, K) \to \mathrm{Hom}(\Gamma, K)$$

is a homeomorphisms onto those components of the target intersecting its image and, consequently, that the same holds for their composition. $\square$

Proof of Theorem 1.1. The theorem follows at once by [6, Theorem I]. $\square$

We can now prove:

Corollary 3.2. *If Γ and K are as in* Proposition 3.1*, then there is a homeomorphism*

$$\mathrm{Hom}(\Gamma, K)_{\mathbb{1}} \cong \mathrm{Hom}(\mathbb{Z}^{rank\, H_1(\Gamma;\mathbb{Z})}, K)_{\mathbb{1}}.$$

Proof. By Proposition 3.1, we have a homeomorphism

$$\mathrm{Hom}(\Gamma, K)_{\mathbb{1}} \cong \mathrm{Hom}(H_1(\Gamma;\mathbb{Z}), K)_{\mathbb{1}}.$$

Since $H_1(\Gamma;\mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$ is a finitely generated abelian group, we may identify $H_1(\Gamma;\mathbb{Z})$ with $\mathbb{Z}^r \oplus A$ where $r := \mathrm{rank}\, H_1(\Gamma;\mathbb{Z})$ and A is a finite abelian group. At this point we would like to show that $\mathrm{Hom}(\mathbb{Z}^r \oplus A, K)_{\mathbb{1}} = \mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}$. Seeking a contradiction, suppose that $\rho_0 \in \mathrm{Hom}(\mathbb{Z}^r \oplus A, K)_{\mathbb{1}}$ maps A non-trivially into K. By assumption, there is a continuous path of representations $[0,1] \mapsto \rho_t$ starting

at ρ_0 and ending at the trivial representation $\rho_1 = \mathbb{1}$. But now, this path induces a continuous deformation in $\mathrm{Hom}(A, K)$ of the representation $\rho_0|_A$ to the trivial representation. This is impossible since Lie groups contain no small subgroups. $\square$

Proof of Corollary 1.2. The corollary follows at once by [6, Theorem I]. $\square$

Using this, we immediately obtain:

Corollary 1.3. *Let G be the group of complex points of a reductive algebraic group. If Γ is a finitely generated nilpotent group, then:*

(1) $\pi_1(\mathrm{Hom}(\Gamma, G)_{\mathbb{1}}) \cong \pi_1(G)^{rank\, H_1(\Gamma;\mathbb{Z})}$, *and*
(2) $\pi_1((\mathrm{Hom}(\Gamma, G)/\!\!/G)_{\mathbb{1}}) \cong \pi_1(G/[G, G])^{rank\, H_1(\Gamma;\mathbb{Z})}$.

Proof. The two formulas follow at once from Corollary 1.2 by the main results of Gómez–Pettet–Souto [15] and Biswas–Lawton–Ramras [8]. $\square$

In order to prove our second corollary, we need the following:

Lemma 3.3. *If K is a compact Lie group and Γ is a finitely generated nilpotent group, then* $\mathrm{Hom}(\Gamma, K)_{\mathbb{1}}/K = (\mathrm{Hom}(\Gamma, K)/K)_{\mathbb{1}}$. *In particular,* $\mathrm{Hom}(\Gamma, K)$ *is connected if and only if* $\mathrm{Hom}(\Gamma, K)/K$ *is connected.*

Proof. Recall from Corollary 3.2 that any $\rho \in \mathrm{Hom}(\Gamma, K)_{\mathbb{1}}$ factors through the torsion free part of $H_1(\Gamma; \mathbb{Z})$. As such, by [5, Lemma 4.2], $\rho \in \mathrm{Hom}(\Gamma, K)_{\mathbb{1}}$ if and only if there is a torus $T \subset K$ such that $\rho(\Gamma) \subset T$. Since this property is preserved under conjugation by elements of K, it follows that $(\mathrm{Hom}(\Gamma, K)/K)_{\mathbb{1}}$ coincides with the quotient $\mathrm{Hom}(\Gamma, K)_{\mathbb{1}}/K$. $\square$

We can now prove the cohomological formulas mentioned in the introduction.

Corollary 1.4. *Let G be the group of complex points of a connected reductive algebraic group, let $T \subset G$ be a maximal algebraic torus and let W be the Weyl group of G. If Γ is a finitely generated nilpotent group and F is a field of characteristic* 0 *or relatively prime to the order of W, then:*

(1) $H^*(\mathrm{Hom}(\Gamma, G)_{\mathbb{1}}; F) \cong H^*(G/T \times T^{rank\, H_1(\Gamma;\mathbb{Z})}; F)^W$, *and*
(2) $H^*((\mathrm{Hom}(\Gamma, G)/\!\!/G)_{\mathbb{1}}; F) \cong H^*(T^{rank\, H_1(\Gamma;\mathbb{Z})}; F)^W$.

Proof of Corollary 1.4. Following Pettet–Souto [19, Corollary 1.5], let $K \subset G$ be a maximal compact subgroup such that $T_K := T \cap K$ is a maximal torus in K. Notice that, for any $r \in \mathbb{N}$,

$$K/T_K \times T^r \to G/T \times T^r$$

is a W-equivariant homotopy equivalence and, in particular, that

$$H^*(K/T_K \times T^r)^W \cong H^*(G/T \times T^r)^W. \tag{2}$$

Here, it follows from Baird [5, Theorem 4.3] that the left hand side of the equation is isomorphic to $H^*(\mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}})$. Now, letting $r := \operatorname{rank} H_1(\Gamma; \mathbb{Z})$, our first formula follows at once from the homotopy equivalences

$$\mathrm{Hom}(\Gamma, G)_{\mathbb{1}} \simeq \mathrm{Hom}(\mathbb{Z}^r, G)_{\mathbb{1}} \simeq \mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}$$

provided by Corollary 1.2 and [6, Theorem I]. Finally, it is also due to Baird [5, Remark 4] that $\mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}/K \cong T_K^r/W$ so our second formula follows from the homotopy equivalence and homeomorphisms

$$(\mathrm{Hom}(\Gamma, G)/\!\!/G)_{\mathbb{1}} \simeq (\mathrm{Hom}(\Gamma, K)/K)_{\mathbb{1}} \cong \mathrm{Hom}(\Gamma, K)_{\mathbb{1}}/K \cong \mathrm{Hom}(\mathbb{Z}^r, K)_{\mathbb{1}}/K.$$

provided by [6, Theorem II], Lemma 3.3 and Corollary 3.2. □

Remark. The homotopy types of distinct components of representation spaces are typically different. For instance, if we take Γ to be the discrete Heisenberg group $H_3(\mathbb{Z})$, then $\mathrm{Hom}(\Gamma, \mathrm{SL}_2\,\mathbb{C})$ decomposes into a simply-connected component and a non simply-connected component. In fact, this phenomenon already occurs for Γ abelian as illustrated in Gómez–Adem [2] and Gómez–Pettet–Souto [15].

Theorem 1.5. *Let G be the group of complex points of a reductive algebraic group. If Γ is a finitely generated nilpotent group which surjects onto a finite non-abelian subgroup of G, then $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(\Gamma, G)/\!\!/G$ are both disconnected.*

Proof. Let $\psi : \Gamma \to N$ be a surjective homomorphism onto a finite non-abelian subgroup of G and let K be a maximal compact subgroup of G containing N. Notice in particular that $\psi \in \mathrm{Hom}(\Gamma, K) \subset \mathrm{Hom}(\Gamma, G)$. Since $\mathrm{Hom}(\Gamma, K) \simeq \mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(\Gamma, K)/K \simeq \mathrm{Hom}(\Gamma, G)/\!\!/G$ by [6], and since $\mathrm{Hom}(\Gamma, K)$ is disconnected if and only if $\mathrm{Hom}(\Gamma, K)/K$ is disconnected by Lemma 3.3, it suffices to prove that $\mathrm{Hom}(\Gamma, K)$ is disconnected.

Seeking a contradiction, suppose that $\mathrm{Hom}(\Gamma, K)$ is connected and recall from Proposition 3.1 that, in this case, $\mathrm{Hom}(\Gamma/\Gamma_{(i)}, K)$ is connected for all $i \geq 2$. Choose a minimal $s \in \mathbb{N}$ with the property that $\psi(\Gamma_{(s+1)}) = e_K$ and denote the s-step nilpotent group $\Gamma/\Gamma_{(s+1)}$ by $\hat{\Gamma}$. If we consider the fibre bundle (c.f. Proposition 2.1)

$$p : \mathrm{Hom}(\hat{\Gamma}, K) \to \mathcal{F},\ p(\rho) = \rho(\hat{\Gamma}_{(s)}),$$

then $p(\psi) = \psi(\hat{\Gamma}_{(s)}) \neq e_K$. As such, by Proposition 3.1 and our assumptions,

$$\psi \notin \mathrm{Hom}(\hat{\Gamma}, K)_{\mathbb{1}} \cong \mathrm{Hom}(\Gamma/\Gamma_{(s+1)}, K)_{\mathbb{1}} \cong \mathrm{Hom}(\Gamma, K)_{\mathbb{1}} \cong \mathrm{Hom}(\Gamma, K)$$

and this contradiction completes the proof. □

Corollary 1.6. *Let Γ be a non-abelian free nilpotent group or a Heisenberg group. If G is the group of complex points of a reductive algebraic group, then $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(\Gamma, G)/\!\!/G$ are connected if and only if G is an algebraic torus.*

Proof. If G is disconnected or not simply-connected then [19, Corollary 1.3], [6, Theorems I and II] and Lemma 3.3 show that $\mathrm{Hom}(H_1(\Gamma;\mathbb{Z}),G)$ and $\mathrm{Hom}(H_1(\Gamma;\mathbb{Z}),G)/\!\!/G$ are disconnected. As such, it suffices to consider the case where G is simply-connected. Notice that such a G contains a subgroup isomorphic to $\mathrm{SL}_2\,\mathbb{C}$ and, since $\mathrm{SL}_2\,\mathbb{C}$ contains a copy of the non-abelian group Q of order 8 generated by the matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so does G. Since Q is a $\mathbb{Z}/2\mathbb{Z}$ central extension of $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, it follows that if Γ is either a non-abelian free nilpotent group or a Heisenberg group, then Γ surjects onto Q. The claim now follows from Theorem 1.5. ☐

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